Lecture 2: Some General Theorems on Surfaces

$$0 \to \mathbb{Z} \to {}^h\mathcal{O}_X \to {}^h\mathcal{O}_X^* \to 0$$

(${}^h\mathcal{O}_X$: sheaf of holo. functions on X)

$$\rightsquigarrow H^1(X,\mathbb{Z}) \hookrightarrow H^1(X, {}^h\mathcal{O}_X) \to H^1(X, {}^h\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z})$$

($H^1(X,\mathbb{Z})$ as a lattice; c_1: taking 1st Chern class)

GAGA: $H^1(X, {}^h\mathcal{O}_X) \cong H^1(X, \mathcal{O}_X)$, $H^1(X, {}^h\mathcal{O}_X^*) \cong H^1(X, \mathcal{O}_X^*)$

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*)$$

$$0 \to H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) \to \mathrm{Pic}(X) \to NS(X) \to 0$$

($H^1(X,\mathcal{O}_X^*) = \mathrm{Pic}(X)$; $NS(X)$: Néron–Severi group generated by effective curves)

Hodge theory $\Rightarrow \operatorname{rk} H^1(X,\mathbb{Z}) = h^1(X,\mathcal{O}_X)$

$\therefore H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$ complex torus parametrizing flat line bundles

$\mathcal{F}$: coherent sheaf on X

$$\chi(\mathcal{F}) := \sum_i (-1)^i \dim_{\mathbb{C}} H^i(X,\mathcal{F})$$

- If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, then $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$
- arithmetic genus $p_a(X) := \chi(\mathcal{O}_X) - 1$

Theorem (Riemann–Roch of Surfaces) X compact complex surface

D = divisor on X

then $$\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = \frac{1}{2} D.(D-K)$$

($D.(D-K)$ must be even!)

pf: $D = \underset{\wr\atop C}{\overset{\text{very ample}}{(D+nH)}} - \underset{\wr\atop C'}{nH}$, H : very ample

$\therefore$ $D \sim C - C'$, C, C' smooth Bertini's theorem

$$0 \to \mathcal{O}_X(C-C') \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)\otimes\mathcal{O}_{C'} \to 0$$

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_X(C)\otimes\mathcal{O}_C \to 0$$

$$\chi(\mathcal{O}_X(C-C')) = -\chi(\mathcal{O}_X(C)\otimes\mathcal{O}_{C'}) + \underline{\chi(\mathcal{O}_X(C))}$$

$$\qquad\qquad\qquad\qquad \| \quad \chi(\mathcal{O}_X(C)\otimes\mathcal{O}_C) - \chi(\mathcal{O}_X)$$

$$\therefore\ \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X) = -\chi(\mathcal{O}_X(C)\otimes\mathcal{O}_{C'}) + \chi(\mathcal{O}_X(C)\otimes\mathcal{O}_C)$$

Riemann-Roch for curves $= (C.C' - \underline{g_{C'} + 1}) + (C^2 - \underline{g_C + 1})$

adjunction $\quad -\frac{1}{2}C'.(C'+K_X) \qquad -\frac{1}{2}C.(C+K_X)$

$$= \frac{1}{2}(C-C').(C-C'-K_X)$$ //

H: ample divisor

Lemma 1: D = divisor s.t $D.H > 0$, $D^2 > 0$

then $\underline{H^0(X, \mathcal{O}_X(nD)) > 0}$, for $n \gg 0$

i.e $nD \sim D' > 0$

pf: • $H^0(X, \mathcal{O}_X(K_X - nD)) = 0$, $n \gg 0$

Otherwise $\exists \underset{\overset{\vee}{0}}{D'} \sim K_X - nD$ $\therefore$ $H.D' = K_X.H - n\underset{\overset{\vee}{0}}{(D.H)}$

• $H^0(X, \mathcal{O}_X(nD)) + H^2(X, \mathcal{O}_X(nD))$ $\to 0$ Serre duality

$= \chi(\mathcal{O}_X) + \frac{1}{2}\, \underline{nD \cdot (nD - K_X)} + H^1(X, \mathcal{O}_X(nD)) > 0$ as $n \gg 0$

($D^2 > 0$; $H^1 \geq 0$)

Theorem 2 (Hodge Index Theorem)

H: ample divisor on X

D: divisor s.t $D.H = 0$, $D \not\equiv 0$ $\Longrightarrow D^2 < 0$

(numerical equivalence)

pf: Assume that $D^2 \geq 0$

If $D^2 > 0$, then $nD \sim D' > 0$ and thus $D'.H > 0$ $\times$

If $D^2 = 0$, $\exists E$ s.t $D.E \neq 0$ $\because D \not\equiv 0$

replace E by $(H^2)E - (E.H)H$

$\propto$ projection of E to $H^\perp$

then can assume that $E.H = 0$, $D.E > 0$

Take $D' = nD + E$, $n \gg 0$

$D'^2 > 0$, $D'.H = 0$ $\Longrightarrow$ contradiction

(previous argument)

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Corollary 1. Signature of induced pairing on $NS(X)$ is $(1, n-1)$. $n = \mathrm{rk}\, NS(X)$.

Corollary 2. C, D nef divisors w/ $C.D = 0$, $C^2 \geq 0$, $D^2 \geq 0$.
then $C = \alpha D$ numerically.

useful in studying linear system

pf: H = ample divisor

Assume that $H.C = \alpha(H.D)$,

then $H.(C - \alpha D) = 0$, $(C - \alpha D)^2 \geq 0$

$C.D = 0$, $C^2 \geq 0$, $D^2 \geq 0$

$$\underset{\text{Hodge index thm}}{\Longrightarrow} \quad C - \alpha D \equiv 0$$

Remark: Notice that the above arguments only require that $H.D > 0$ for effective D. Thus, one can take H to be a Kähler class & the statement holds for Kähler surfaces.

Corollary 3. $f: Y \longrightarrow X$ proper morphism

Y smooth, X normal projective surface

E_i = exceptional divisor

then $(E_i \cdot E_j)$ is negative definite.

pf: $D = \sum a_i E_i \neq 0$. Assume that $D^2 \geq 0$

Write $D = A - B$, A, B effective w/ no common component

$0 \le D^2 = A^2 - 2A.B + B^2 \Longrightarrow A^2 \geqslant 0$ or $B^2 \geqslant 0$

WLOG, say $A^2 \geqslant 0$

H: ample divisor on X

$(f^*H)^2 = H^2 > 0$, $A.f^*H = 0 \longrightarrow A^2 < 0$ or ~~$A = 0$~~ $\times$

Corollary 1

irreducible exceptional divisor self-intersection < 0

Corollary 4. (negativity lemma) useful in minimal model program birational geometry

$f: Y \to X$ proper birational w/ exceptional divisor E_i

If $D = \sum a_i E_i$ f-nef, then $-D$ is effective.

pf: First assume that X, Y are surfaces.

Write $D = A - B$, A, B effective w/ no common components

D f-nef $\Rightarrow$ $0 \le D.A = (A - B).A = A^2 - A.B$

$\underline{A^2 \leqslant 0}$, $A.B \geq 0$ $\overset{\text{Corollary 1}}{\Longrightarrow}$ $A = 0$

Corollary 3

or $-D = B$ effective

In higher dimensional case, cut X by hyperplanes repeatedly to reduce to the surface case.

Theorem 3 (Nakai–Moishezon criterion)

A divisor D on X (projective) is ample iff $D^2 > 0$, $D.C > 0$ for all irreducible curve C.

pf: ($\Rightarrow$) obvious since mD is very ample, $m \gg 0$.

($\Leftarrow$) Let H be an very ample divisor represented by an irreducible curve (by Bertini's theorem)

$\Longrightarrow$ (assumption) $D.H > 0$

$\Longrightarrow$ (Lemma 1) nD effective, $n \gg 0$ (replace D)

Let $\mathcal{L} = \mathcal{O}_X(D)$.

Claim: $\mathcal{L}$ is generated by global sections

Since $\mathcal{L}$ is generated by global sections outside D, it suffices to prove that $\mathcal{L} \otimes \mathcal{O}_D$ is semi-ample by Nakayama's lemma.

$\mathcal{L} \otimes \mathcal{O}_D$ ample $\Longleftrightarrow$ $\mathcal{L} \otimes \mathcal{O}_{D_{red}}$ ample

$\Longleftrightarrow$ $\mathcal{L} \otimes \mathcal{O}_{D_i}$ ample, D_i irreducible component of D_{red}

$\Longleftrightarrow f_i^*(\mathcal{L}\otimes\mathcal{O}_{D_i})$ ample . $f_i : \tilde{D}_i \to D_i$ normalization

$\Longleftrightarrow \deg_{\tilde{D}_i} f_i^*(\mathcal{L}\otimes\mathcal{O}_{D_i}) > 0$

$\parallel$

$\deg_{D_i} \mathcal{L}\otimes\mathcal{O}_{D_i} = \underline{D.D_i > 0}$

D_i irreducible , assumption

$$0 \to \mathcal{L}^{-1} \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \qquad \otimes \mathcal{L}^n$$

$$\rightsquigarrow \quad 0 \to H^0(X,\mathcal{L}^{n-1}) \to H^0(X,\mathcal{L}^n) \to H^0(X,\mathcal{L}^n\otimes\mathcal{O}_D)$$

$$\to H^1(X,\mathcal{L}^{n-1}) \to H^1(X,\mathcal{L}^n) \to \underline{H^1(X,\mathcal{L}^n\otimes\mathcal{O}_D)} = 0$$

$n \gg 0$, $\because \mathcal{L}\otimes\mathcal{O}_D$ ample

$\dim_{\mathbb{C}} H^1(X,\mathcal{L}^n) \searrow$ thus stablize. as $n \gg 0$

and $H^0(X,\mathcal{L}^n) \twoheadrightarrow H^0(X,\mathcal{L}^n\otimes\mathcal{O}_D)$

Thus, $\mathcal{L}$ is generated by global sections.

$$\rightsquigarrow \varphi : X \xrightarrow{|\mathcal{L}|} \mathbb{P}^N$$

Claim: fibre of φ is finit points

Otherwise $\exists$ $C \subseteq X$ s.t $\varphi(C) =$ point
irreducible
curve

$\Longrightarrow \mathcal{L}\otimes\mathcal{O}_C$ trivial or $D.C = 0$ $\times$

Stein factorization $\longrightarrow$ φ finite morphism

$f: X \longrightarrow Y$ projective Noetherian

projective morphism w/ finite fibres is finite

$\Rightarrow$ $\exists Z$ with $X \dashrightarrow Z$ projective w/ connected fibres, and $Z \dashrightarrow Y$ finite

$\Longrightarrow$ $\mathcal{L}$ is ample

$\varphi: X \longrightarrow Y$ finite surjective

$\mathcal{L}$ ample on Y iff $\varphi^*\mathcal{L}$ ample on X

Remark: It is stronger than directly applying Kleiman's criterion of ampleness, which also needs positivity on $\overline{NE}(X)$.